Def. La v. a. X si dice assolutamente continua se esiste una funzione $f : \mathbb{R} \to \mathbb{R}^+$ integrabile su $\mathbb{R}$ e tale che, $\forall A$ intervallo in $\mathbb{R}$, si abbia

$$P(X \in A) = \int_A f(x)\,dx$$

f si chiama densità di X.

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

Viceversa, se f è una funzione $: \mathbb{R} \to \mathbb{R}^+$ integrabile su $\mathbb{R}$ e tale che

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1,$$

allora esiste una v. a. X che ammette f come sua densità

Se X ammette densità f, allora

$$P(X \le t) = \int_{-\infty}^{t} f(x)\,dx$$

$$p(x) = P(X = x)$$

Caso discreto	Caso ass. cont.
1) $P(X\in A)=\sum'_{x\in A} p(x)$	1) $P(X\in A)=\int_A f(x)\,dx$
2) X ha sper. finita se $\sum'_{x\in\mathbb{R}} \lvert x\rvert\, p(x)<\infty$	2) X ha sper. finita se $\int_{-\infty}^{+\infty} \lvert x\rvert f(x)\,dx<\infty$
3) $E[X]=\sum'_{x\in\mathbb{R}} x\,p(x)$	3) $E[X]=\int_{-\infty}^{+\infty} x\, f(x)\,dx$
4) $E[\varphi(X)]= =\sum'_{x\in\mathbb{R}} \varphi(x)\,p(x)$	$E[\varphi(X)]= \int_{-\infty}^{+\infty}\varphi(x)\,f(x)\,dx$
5) $E[X^k]= \sum_{x\in\mathbb{R}} x^k p(x)$	5) $E[X^k]= =\int_{-\infty}^{+\infty} x^k f(x)\,dx$
6) $\mathrm{Var}\,X=E\left[\left(X-E[X]\right)^2\right] =\sum'_{x\in\mathbb{R}}\left(x-E[X]\right)^2 p(x)$	$\mathrm{Var}\,X=E[(X-E[X])^2]= =\int_{-\infty}^{+\infty}\left(x-E[X]\right)^2 f(x)\,dx$
7) $\mathrm{Var}\,X=E[X^2]-E^2[X] =\sum'_{x\in\mathbb{R}} x^2 p(x)-\left(\sum'_{x\in\mathbb{R}} x\,p(x)\right)^2$	7) $\mathrm{Var}\,X=E[X^2]-E^2[X] =\int_{-\infty}^{+\infty} x^2 f(x)\,dx-\left(\int_{-\infty}^{+\infty} x f(x)\,dx\right)^2$

Principali densità (caso ass. cont.)

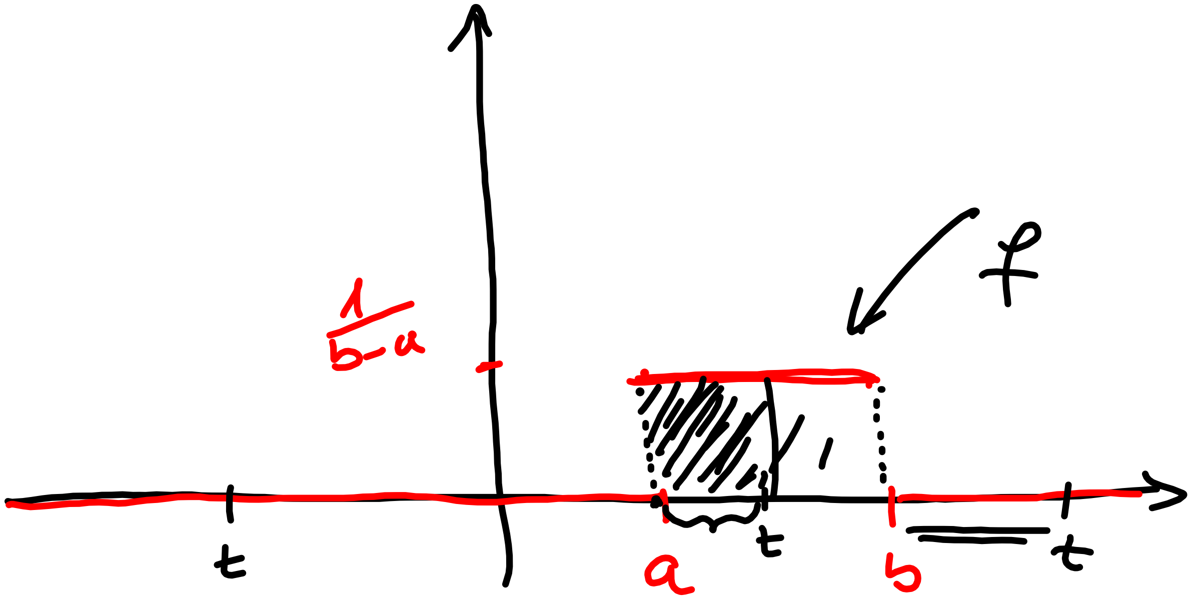
1) Legge uniforme su (a,b) (intervallo <u>limitato</u>)

X ha densità uniforme su (a,b)

$\left(X \sim \mathcal{U}((a,b)) \right)$ se

ammette la seg. densità

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{se } a < x < b \\ 0 & \text{altrove} \end{cases}$$



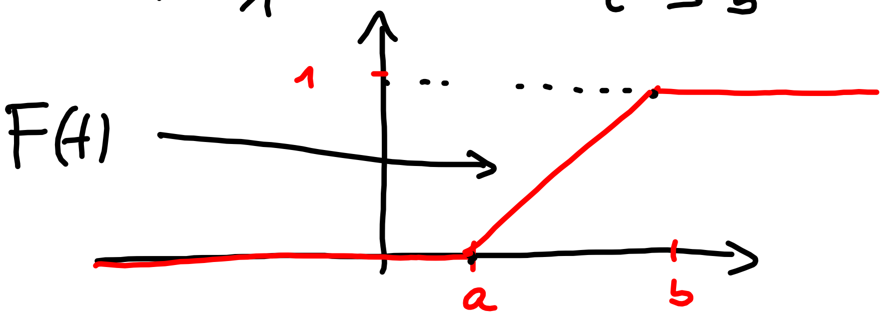
$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} 0\,dx + \int_{a}^{b} \frac{1}{b-a}\,dx + \int_{b}^{+\infty} 0\,dx$$

$$= \frac{1}{b-a} \cdot (b-a) = 1$$

$$F(t) = \int_{-\infty}^{t} f(x)\,dx =$$

$$= \begin{cases} 0 & t < a \\ \cancel{\int_{-\infty}^{a}} + \int_a^t \frac{1}{b-a}\,dx = \frac{t-a}{b-a} & a \leq t \leq b \\ \cancel{\int_{-\infty}^{a}} + \int_a^b + \cancel{\int_b^t} = 1 & t > b \end{cases}$$

$$F(t) = \begin{cases} 0 & t < a \\ \dfrac{t-a}{b-a} & a \leq t \leq b \quad \leftarrow \\ 1 & t > b \end{cases}$$



$$P(u \le X \le v) = \underline{P(X \le v) - P(X < u)}$$

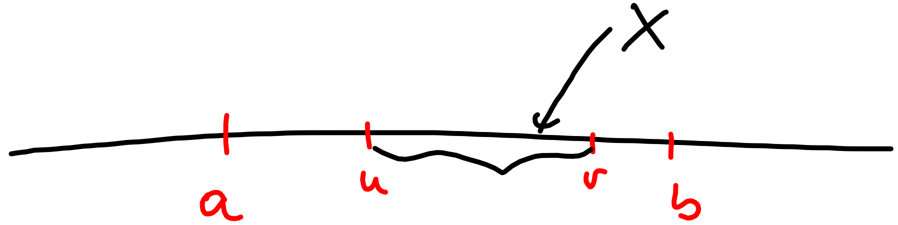
$$\boxed{\lim_{t \to v^-} F(t) - F(u)} = F(v) - F(u)$$

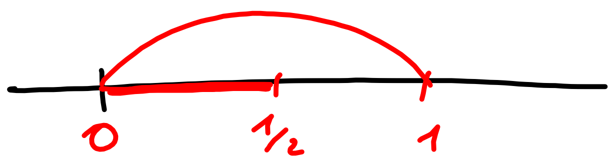
$$F(v) - \lim_{t \to u^-} F(t) = F(v) - F(u)$$

$$P(u < X < v) = F(v) - F(u)$$

$a \le u < v \le b$

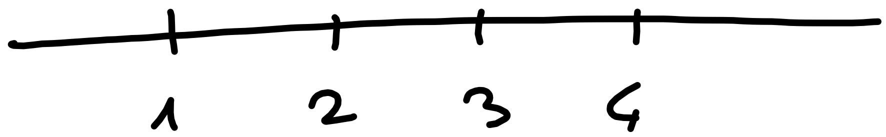
$$= \frac{v-a}{b-a} - \frac{u-a}{b-a} = \frac{v-u}{b-a}$$

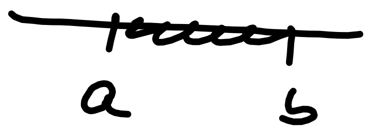




$P(0 < X \le \frac{1}{2})$

Si sceglie un punto "a caso in modo uniforme" nell'intervallo (a, b)



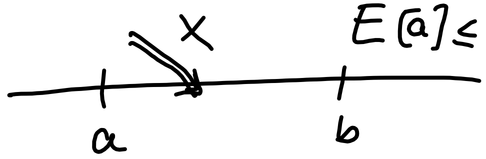


Sia X una v.a. con densità $\mathcal{U}([a,b])$. Calcolare $E[X]$

$$\int_{-\infty}^{+\infty} |x| \, \underline{f(x)} \, dx < +\infty$$

$$= \int_a^b |x| \frac{1}{b-a} \, dx$$

$a \leq X \leq b$



$E[a] \leq \underline{\underline{E[X]}} \leq \underline{E[b]}$

$X \leq Y$

$$a \leq \underline{E[X]} \leq b$$

$$P(X \in (a,b)) = 1$$

$$= \int_a^b f(x) \, dx = \int_a^b \frac{1}{b-a} \, dx = 1$$

$$P(X \in A) = \int_A f(x) \, dx$$

$$X \sim f(x) = \begin{cases} h(x) & x \in I \\ \underline{0} & \underline{x \notin I} \end{cases}$$

allora $P(X \in I) = 1$

$$P(X \in I) = \int_I f(x)\,dx =$$

$$= \int_{\mathbb{R}} f(x)\,dx = \underline{1}$$

$$X \sim f(x) = \begin{cases} \frac{1}{b-a} & \underbrace{a < x < b}_{} \\ 0 & \text{---} \end{cases}$$

$$P(X \in (a,b)) = 1$$

$$P(a \leq X \leq b) = 1$$

$$a \leq E[X] \leq b$$

$$E[X] = \int_{-\infty}^{+\infty} x \, \underline{f(x)} \, dx =$$

$$= \int_{-\infty}^{a} 0 \, dx + \int_{a}^{b} x \frac{1}{b-a} dx + \int_{b}^{+\infty} 0 \, dx =$$

$$= \frac{1}{b-a} \int_{a}^{b} x \, dx = \frac{1}{b-a} \left. \frac{x^2}{2} \right|_{a}^{b} =$$

$$= \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{b+a}{2}$$

$$\mathrm{Var}\, X = E[X^2] - E^2[X] =$$

$$= E[X^2] - \left(\frac{a+b}{2}\right)^2 = *$$

$$E[X^2] = E[\varphi(X)] =$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)\, dx =$$

$$= \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{1}{b-a} \left.\frac{x^3}{3}\right|_a^b$$

$$= \frac{b^3 - a^3}{3} \cdot \frac{1}{b-a} = \frac{b^2+ab+a^2}{3}$$

$$* = \frac{b^2+ab+a^2}{3} - \frac{a^2+2ab+b^2}{4} =$$

$$= \frac{b^2+a^2-2ab}{12} = \frac{(b-a)^2}{12}$$

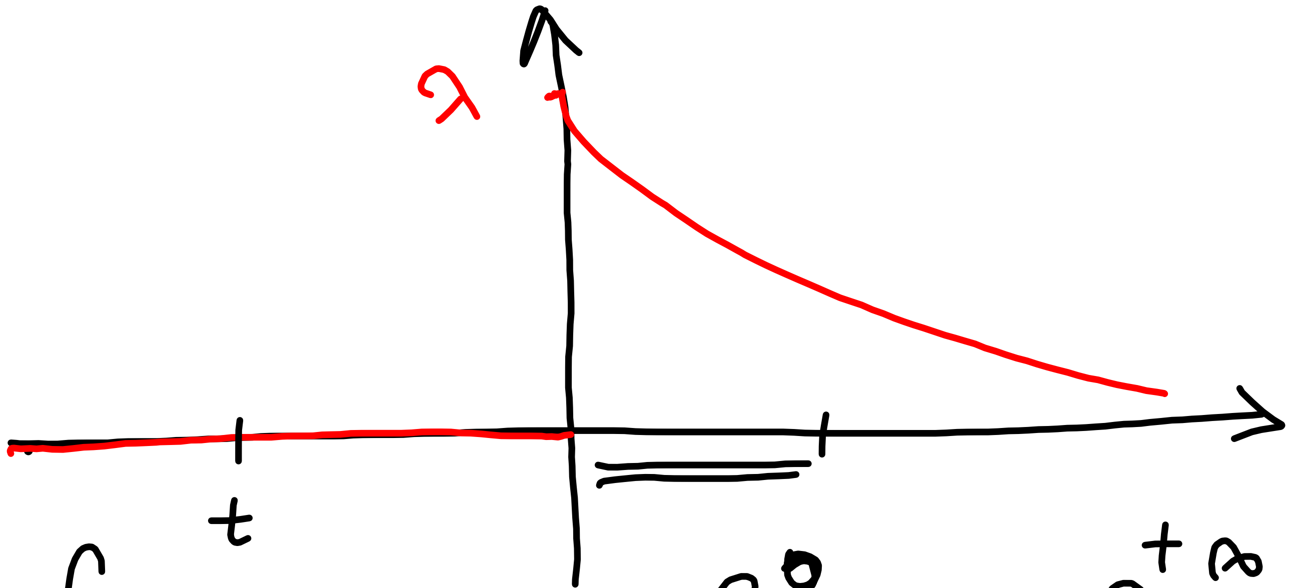
2) Legge esponenziale $\lambda > 0$

$\mathcal{E}(\lambda)$ –

$X \sim \mathcal{E}(\lambda)$ se ha la seguente densità

$$X \sim f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$P(X \geq 0) = 1$$



$$\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{+\infty} \lambda e^{-\lambda x}\,dx =$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{+\infty} = \lambda \left(0 - \frac{1}{-\lambda} \right)$$

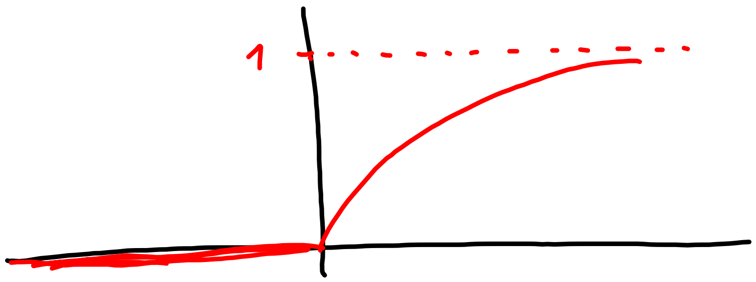
$$= \lambda \cdot \frac{1}{\lambda} = 1$$

$$F(t) = \int_{-\infty}^{t} f(x)\,dx =$$

$$= \begin{cases} 0 & t < 0 \\ \int_0^t \lambda e^{-\lambda x} dx & t > 0 \end{cases} =$$

$$\lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^t = \lambda \left[\frac{e^{-\lambda t} - 1}{-\lambda} \right] = \underline{1 - e^{-\lambda t}}$$

$$= \begin{cases} 0 & t < 0 \\ 1 - e^{-\lambda t} & t \geq 0 \end{cases}$$



X = durata dello strumento

Sia $X \sim \mathcal{E}(\lambda)$. Allora

$$\to P(X > t+s \mid X > t) = P(X > s) \qquad \begin{matrix} t>0 \\ s>0 \end{matrix}$$

Proprietà di assenza di usura
assenza di memoria

$$P(X>t+s \mid X>t) = \frac{P(X>t+s,\ X>t)}{P(X>t)} =$$

$$= \frac{P(X>t+s)}{P(X>t)} =$$

$$= \frac{1-\overbrace{P(X\le t+s)}^{F(t+s)}}{1-\underbrace{P(X\le t)}_{F(t)}} = \frac{1-(1-e^{-\lambda(t+s)})}{1-(1-e^{-\lambda t})}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X>s)$$

$$= 1-(1-e^{-\lambda s})$$

$$E[|X|] = \int_{-\infty}^{+\infty} |x| \, f(x)\,dx = \int_{0}^{+\infty} |x| \lambda e^{-\lambda x} dx$$

$$E[X] = \int_{0}^{+\infty} x \lambda e^{-\lambda x} dx$$

$$E[X] = \lambda \int_{0}^{+\infty} x \underbrace{e^{-\lambda x}} dx =$$

$$= \lambda \left[\left. \frac{e^{-\lambda x}}{-\lambda} x \right|_{0}^{+\infty} - \int_{0}^{+\infty} \frac{e^{-\lambda x}}{-\lambda} dx \right] =$$

$$= \lambda \left[\cancel{[0 - 0.]} + \frac{1}{\lambda} \int_{0}^{+\infty} e^{-\lambda x} dx \right]$$

$$\mathrm{Var}\,X = \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda} \boxed{\int_{0}^{+\infty} \lambda e^{-\lambda x} dx} = \frac{1}{\lambda} = E[X]$$

Densità Gamma $\alpha > 0$ $\lambda > 0$

$X \sim \Gamma(\alpha, \lambda)$

$\Gamma(\alpha)$ (Gamma di Eulero)

$$\Gamma(\alpha) = \int_0^{+\infty} \underline{\underline{x}}^{\alpha-1} e^{-x} \, dx < \infty \quad \forall \alpha > 0$$

$$\alpha = 0 \quad \int_{\underline{0}}^{+\infty} \frac{1}{\underline{x}} e^{-x} \, dx \; ?$$

$$\int_0^1 \frac{1}{x} \, dx = \log x \Big|_0^1 = \log 1 - \log 0$$

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} \, dx \qquad \alpha > 0$$

$$\Gamma(1) = \int_0^{+\infty} e^{-x} \, dx = 1$$

$$\Gamma(2) = \int_0^{+\infty} \underbrace{x}\,\underbrace{e^{-x}} \, dx = 1$$

$$\Gamma(300) = \int_0^{+\infty} \underbrace{x^{299}}\,\underbrace{e^{-x}} \, dx$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

per parti

$$= \int_0^{+\infty} x^{\alpha} e^{-x}\,dx =$$

$$= \cancel{\left. -e^{-x} x^{\alpha} \right|_0^{+\infty}} - \int_0^{+\infty} -e^{-x} \alpha x^{\alpha-1}\,dx$$

$$= +\alpha \int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx = \alpha\,\Gamma(\alpha)$$

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$$

$$\Gamma(2) = 1\,\Gamma(1) = 1 \cdot 1 = 1!$$

$$\Gamma(3) = 2\,\Gamma(2) = 2 \cdot 1 = 2!$$

$$\Gamma(4) = 3\,\Gamma(3) = 3 \cdot 2 \cdot 1 = 3!$$

$$\Gamma(5) = 4\,\Gamma(4) = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$$

$$\vdots$$

$$\boxed{\Gamma(n) = (n-1)!} \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\int_0^{+\infty} x^{n-1} e^{-x}\,dx$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Si dice che $X \sim \Gamma(\alpha, \lambda)$ se ammette la seg. densità $\quad \mathcal{E}(\lambda) = \Gamma(1, \lambda)$

$$f(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$\alpha = 1 \quad \lambda e^{-\lambda x}$

$\lambda x = y$

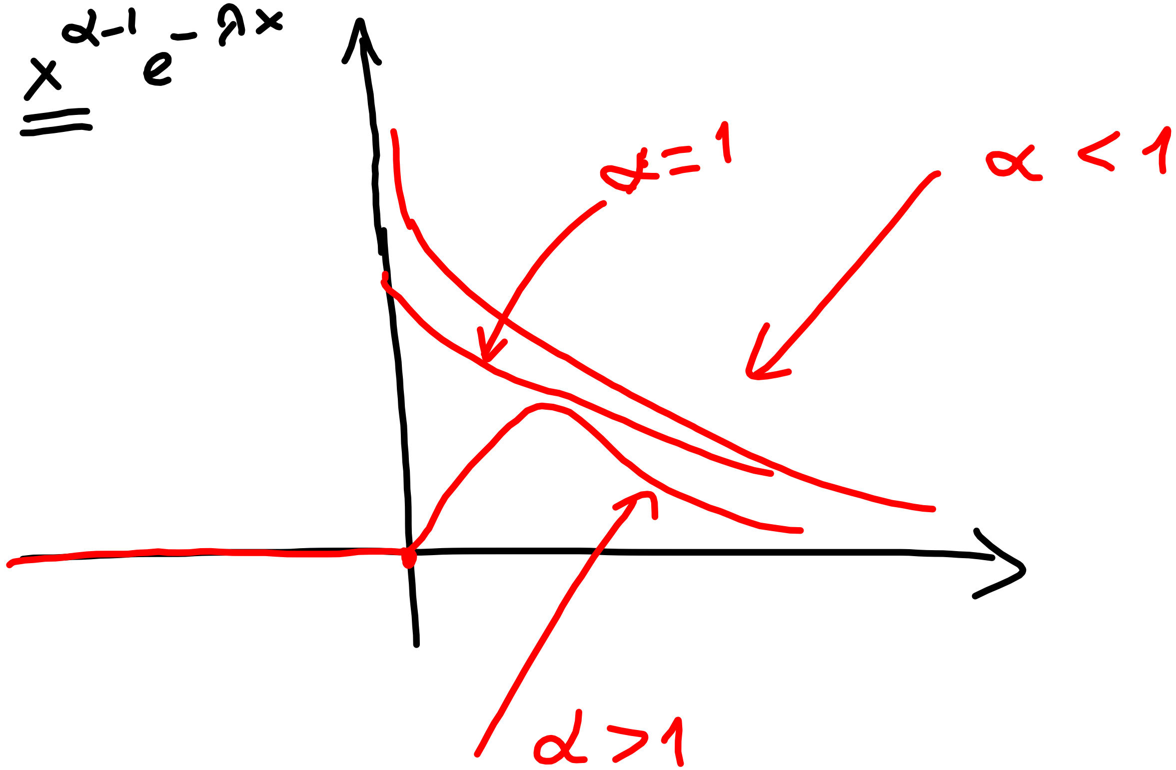
$dx = \frac{dy}{\lambda}$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\,dx$$

$$= \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-y} \frac{dy}{\lambda} = \int_0^{+\infty} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} y^{\alpha-1} e^{-y} dy = \frac{\Gamma(\alpha)}{\Gamma(\alpha)} = 1$$

$$\frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$



$$E[X] = \int_{-\infty}^{+\infty} x f(x)\, dx = \int_0^{+\infty} x \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx =$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\lambda x} dx = \frac{\lambda^\alpha}{\cancel{\Gamma(\alpha)}} \int x^{\alpha+1-}$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \underbrace{\boxed{\int_0^{+\infty} \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1} e^{-\lambda x} dx}}_{=1}$$

$$= \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} = \frac{\alpha \Gamma(\alpha)}{\lambda \Gamma(\alpha)} = \frac{\alpha}{\lambda} = E[X]$$

$$\mathrm{Var}\, X = E[X^2] - E^2[X] =$$

$$= \underbrace{\int_0^{+\infty} x^2 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx}_{} - \frac{\alpha^2}{\lambda^2} =$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+2)}{\lambda^{\alpha+2}} \underbrace{\int_0^{+\infty} \frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1} e^{-\lambda x} dx}_{=1} - \frac{\alpha^2}{\lambda^2}$$

$$= \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{(\alpha+1)\Gamma(\alpha+1)}{\Gamma(\alpha)\lambda^2} - \frac{\alpha^2}{\lambda^2} =$$

$$= \frac{(\alpha+1)\alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha(\alpha+1) - \alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

Teorema Siano $\left.\begin{array}{l} X \sim \Gamma(\alpha_1, \lambda) \\ Y \sim \Gamma(\alpha_2, \lambda) \end{array}\right\}$ indipendenti

Allora la v. a.

$$Z = X + Y \sim \Gamma(\alpha_1 + \alpha_2, \lambda)$$

$$\left.\begin{array}{l} X \sim \Gamma(\alpha_1, \lambda) \\ Y \sim \Gamma(\alpha_2, \lambda) \\ Z \sim \Gamma(\alpha_3, \lambda) \end{array}\right\} \text{indipendenti}$$

$$\underbrace{(X+Y)}_{\sim \Gamma(\alpha_1+\alpha_2, \lambda)} + \underbrace{Z}_{\sim \Gamma(\alpha_3, \lambda)} \sim \Gamma(\alpha_1 + \alpha_2 + \alpha_3, \lambda)$$

$\boxed{X+Y}$ $\boxed{Z}$ sono indip ??

$$F(t) = \int_{-\infty}^{t} f(x)\,dx$$

$$\lim_{t \to x_0} \int_{-\infty}^{t} f(x)\,dx = \int_{-\infty}^{x_0} f(x)\,dx$$

Se X è assolutamente continua,
allora X è continua